2.3 The Cauchy Integral Formula

In order to obtain Cauchy's integral formula, a key result of complex analysis, we need a sharpened version of Proposition 2.14.

Proposition 2.16. *Let $D \subset \mathbb{C}$ be a region and $p \in D$. Let $f : D \to \mathbb{C}$ be continuous function which is moreover holomorphic in $D \setminus \{p\}$. Let Δ be a triangle in D such that p is one of the corners of Δ. Then,*

$$\int_{\partial\Delta} f = 0.$$

Proof. Fix $\epsilon > 0$. Denote the corner points of Δ by (p, x, y). Define the triangle Δ_t for $t \in [0, 1]$ as the triangle with corner points (p, x_t, y_t), where $x_t := p + t(x - p)$ and $y_t := p + t(y - p)$. Then, $l(\partial\Delta_t) \to 0$ as $t \to 0$. By continuity of f on the compact set Δ, Proposition 2.7 implies that there exists $t > 0$ such that

$$\int_{\partial\Delta_t} f < \epsilon.$$

Now, subdivide the triangle Δ into the triangle Δ_t and the triangles with corners given by (x_t, x, y) and (x_t, y, y_t). The integral over boundary paths of the latter two triangles vanishes by Proposition 2.14. On the other hand, the sum of the integrals over the boundary paths of the three triangles equals the integral over the boundary path of Δ. Thus,

$$\int_{\partial\Delta} f = \int_{\partial\Delta_t} f < \epsilon.$$

Since ϵ was arbitrary the statement follows. □

Exercise 15. The above Proposition can be strengthened considerably. Show the following: Let $\Delta \subset \mathbb{C}$ be a triangle and let $f : \Delta \to \mathbb{C}$ be continuous. Furthermore, assume that f is holomorphic in the interior of Δ. Then,

$$\int_{\partial\Delta} f = 0.$$

The above proposition implies a corresponding stronger version of Corollary 2.15.

Corollary 2.17. *Let $D \subseteq \mathbb{C}$ be a star-shaped region with center $z_0 \in D$ and $f : D \to \mathbb{C}$ continuous. Furthermore assume that f is holomorphic in $D \setminus \{z_0\}$. Then, f is integrable in D.*

Proof. Combine Proposition 2.13 with Proposition 2.16. □

Definition 2.18. Let γ be a closed path. Let $z \in \mathbb{C} \setminus |\gamma|$ and define the *index* of z with respect to γ as,

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\zeta - z} \,\mathrm{d}\zeta.$$

Theorem 2.19. *Let γ be a closed path and $U := \mathbb{C} \setminus |\gamma|$. Then, $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$ for all $z \in U$. Moreover, $\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_\gamma(z')$ if z and z' are in the same connected component of U. Also, $\mathrm{Ind}_\gamma(z) = 0$ if $|z|$ is sufficiently large.*

Proof. Parametrizing $\gamma : [a, b] \to \mathbb{C}$ we have,

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} \,\mathrm{d}t.$$

In order to show that $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$ we define $\phi : [a, b] \to \mathbb{C}$ via,

$$\phi(t) := \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \,\mathrm{d}s\right).$$

It is then sufficient to show that $\phi(b) = 1$, which we proceed to do. Observe that ϕ is continuous and piecewise continuously differentiable with piecewise differential

$$\phi'(t) = \frac{\phi(t)\gamma'(t)}{\gamma(t) - z}.$$

The quotient function $t \mapsto \phi(t)/(\gamma(t) - z)$ is also continuous and piecewise continuously differentiable with piecewise differential given by,

$$\left(\frac{\phi(t)}{\gamma(t) - z}\right)' = 0.$$

Thus, this function is piecewise constant and continuous. So it must be constant on the connected set $[a, b]$. Equating its value at a with its value at b yields,

$$\phi(b) = \phi(a)\frac{\gamma(b) - z}{\gamma(a) - z} = 1,$$

since $\phi(a) = \exp(0) = 1$ and γ is closed.

Exercise. Show that $\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_\gamma(z')$ if z and z' are in the same connected component of U. [Hint: Show first that $\mathrm{Ind}_\gamma : U \to \mathbb{C}$ is continuous.]

It remains to show that $\mathrm{Ind}_\gamma(z) = 0$ if z is sufficiently large. Let $M := \sup_{t\in[a,b]} |\gamma(t)|$ and $N := \sup_{t\in[a,b]} |\gamma'(t)|$. Then, if $|z| > M + l(\gamma)N$ we have, using Proposition 2.7,

$$|\mathrm{Ind}_\gamma(z)| \le \frac{l(\gamma)N}{|z| - M} < 1.$$

On the other hand $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$, so we must have in this case $\mathrm{Ind}_\gamma(z) = 0$. This completes the proof. $\square$

Exercise 16. Let $\gamma : [0, 1] \to \mathbb{C}$ be the path $\gamma(t) := z_0 + re^{2\pi i k t}$ with $z_0 \in \mathbb{C}$ and $r > 0$ and $k \in \mathbb{Z}$. Show that $\mathrm{Ind}_\gamma(z) = k$ if $z \in B_r(z_0)$ and $\mathrm{Ind}_\gamma(z) = 0$ if $z \in \mathbb{C} \setminus \overline{B_r(z_0)}$.

Theorem 2.20 (Cauchy Integral Formula)**.** *Let $D \subseteq \mathbb{C}$ be a star-shaped region with center z, $f \in \mathcal{O}(D)$, γ a closed path in $D \setminus \{z\}$. Then,*

$$f(z)\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Proof. Define the function $g : D \to \mathbb{C}$ as follows,

$$g(\zeta) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \neq z \\ f'(z) & \text{if } \zeta = z \end{cases}.$$

By the property of the complex derivative of f at z, g is continuous in all of D. Moreover, g is holomorphic in $D \setminus \{z\}$. So, by Corollary 2.17, g is integrable in D. By Proposition 2.11 this implies,

$$0 = \int_\gamma g = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z)\,2\pi i\,\mathrm{Ind}_\gamma(z).$$

$\square$

Let B be an open disk in $\mathbb{C}$. We denote by ∂B its boundary, i.e., $\partial B = \overline{B} \setminus B$. We also denote by ∂B a closed path that traces the boundary ∂B once with positive (anti-clockwise) direction. If B has center z_0 and radius r, the path ∂B can be represented by the corresponding path γ of Exercise 16 with $k = 1$.

The Cauchy Integral Formula is often used in the special case where the path is the boundary of a disk: Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$, $z \in D$ and $r > 0$ such that $\overline{B_r(z)} \subset D$. Then,

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Lemma 2.21. *Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuous and γ a closed path in U. Define the function $F : \mathbb{C} \setminus |\gamma| \to \mathbb{C}$ via*

$$F(z) := \int_\gamma \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta.$$

Then, F is analytic in $\mathbb{C} \setminus |\gamma|$. Moreover, for all $n \in \mathbb{N}_0$,

$$F^{(n)}(z) = n! \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \,\mathrm{d}\zeta.$$

Proof. Fix $z_0 \in \mathbb{C} \setminus |\gamma|$ and define for all $n \in \mathbb{N}_0$,

$$c_n := \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \,\mathrm{d}\zeta.$$

Set $r := \inf_{t\in[a,b]} |\gamma(t) - z_0|$. We proceed to show that the power series

$$G(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

converges in $B_r(z_0)$ and agrees there with $F(z)$. Fix $z \in B_r(z_0)$. Define the partial sums $g_n : |\gamma| \to \mathbb{C}$ for $n \in \mathbb{N}_0$ via,

$$g_n(\zeta) := \sum_{k=0}^{n} \frac{f(\zeta)(z - z_0)^k}{(\zeta - z_0)^{k+1}}.$$

Since $|\zeta - z_0| \geq r > |z - z_0|$ and f is bounded on $|\gamma|$, the sequence of functions $\{g_n\}_{n\in\mathbb{N}_0}$ converges uniformly on $|\gamma|$. Thus, by Proposition 2.8,

$$G(z) = \lim_{n\to\infty} \int_\gamma g_n(\zeta) \,\mathrm{d}\zeta = \int_\gamma \lim_{n\to\infty} g_n(\zeta) \,\mathrm{d}\zeta.$$

In particular, $G(z)$ is well defined and its radius of convergence is at least r. Consider now the identity

$$\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k,$$

for $x \in B_1(0) \subset \mathbb{C}$. Inserting $x = (z - z_0)/(\zeta - z_0)$ and dividing by $(\zeta - z_0)$ we get,

$$\frac{1}{\zeta - z} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.$$

This implies,

$$\lim_{n\to\infty} g_n(\zeta) = \frac{f(\zeta)}{\zeta - z},$$

and hence $G(z) = F(z)$.

Finally, Theorem 1.16 tells us that F is holomorphic and its complex derivative is again analytic in the same region. Iterating the formula from this Theorem for the derivative yields,

$$F^{(n)}(z) = n!\, c_n,$$

and thus the stated formula.

□

Theorem 2.22 (Cauchy-Taylor Representation Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$. Then, f is analytic in D. Moreover, for any $z_0 \in D$ and $r > 0$ such that $\overline{B_r(z_0)} \subseteq D$ we have,*

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta.$$

for all $z \in B_r(z_0)$.

Proof. Fix $z_0 \in D$ and $\rho > 0$ such that $B_\rho(z_0) \subseteq D$. Then choose r such that $0 < r < \rho$. This implies, $\overline{B_r(z_0)} \subset D$ and by Theorem 2.20 and Exercise 16 we have,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

for $z \in B_r(z_0)$. Lemma 2.21 then tells us that f is analytic in $B_r(z_0)$ and that

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta.$$

for $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$. But since r can be chosen arbitrarily close to ρ, the radius of convergence of the power series for f around z_0 is actually at least ρ. Thus, f is analytic in D. This completes the proof. □

This Theorem finally yields the remarkable result that holomorphic functions are analytic. Together with Theorem 1.16 this means that the properties of holomorphicity and analiticity are really equivalent. Furthermore, it implies that the derivative of a holomorphic function is again a holomorphic function.

Corollary 2.23. *Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ integrable. Then $f \in \mathcal{O}(D)$.*

Proof. If f is integrable in D, then there exists a primitive $F \in \mathcal{O}(D)$ of f. But F being holomorphic, its derivative $F' = f$ is also holomorphic by Theorem 2.22. □

Definition 2.24. Let $D \subseteq \mathbb{C}$ be a region. We call $f : D \to \mathbb{C}$ *locally analytic* iff for every point $z \in D$ there is $r > 0$ so that f can be represented by a power series around z with radius of convergence r.

Definition 2.25. Let $D \subseteq \mathbb{C}$ be a region. We call $f : D \to \mathbb{C}$ *locally integrable* iff for every point $z \in D$ there is a neighborhood $U \subseteq D$ of z such that f is integrable in U.

We wrap up this section with the following summary result.

Theorem 2.26. *Let $D \subseteq \mathbb{C}$ be a region. For a function $f : D \to \mathbb{C}$ the following statements are equivalent:*

1. *f is holomorphic in D.*
2. *f is analytic in D.*
3. *f is locally analytic in D.*
4. *f is locally integrable in D.*

Proof. **Exercise.** □

Exercise 17. Calculate the following integrals. [Hint: Use the Cauchy Integral formula]

1. $$\int_{\partial B_2(0)} \frac{e^z}{(z+1)(z-3)^2}\,\mathrm{d}z$$

2. $$\int_{\partial B_2(-2\mathrm{i})} \frac{1}{z^2+1}\,\mathrm{d}z$$

Exercise 18. Determine all entire functions $f \in \mathcal{O}(\mathbb{C})$ which satisfy the differential equation $f'' + f = 0$.